Content Vectors and the Young Graph; Continuation of Analysis of $Spec(n)$

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Abstract

We continue to pursue our goal to describe the set $Spec(n)$ and the equivalence relation defined on it. First, we will define $Cont(n)$ — the set of content vectors of length n — under the motivation to further restrict possible vectors that can be in $Spec(n)$. We will then define the Young graph and some related notions, and show that there is a bijection between $Cont(n)$ and the set of Young tableaux $Tab(n)$ which also preserves the equivalence class in each set. This discussion will not only enable us to fulfill our goal stated above but also help to obtain an explicit model of representations of S_n and derive the formula for their characters.

Contents

1 Review of the Plan

1.1 Basic Definitions

Definition 1.1. The **Young-Jucys-Murphy elements** $X_1, X_2, \ldots, X_n$, or YJM-elements, are the elements of $\mathbb{C}[S_n]$:

$$X_i = (1, i) + (2, i) + \cdots + (i - 1, i)$$

In particular, $X_1 = 0$.

Definition 1.2. The **Coxeter generators** $s_1, ..., s_{n-1}$ are the elements of S_n:

$$s_i = (i, i + 1)$$

Definition 1.3. The **Young Basis** $\mathscr{Y}$ is the union of all Gelfand-Tsetlin basis of all irreducible representations of S_n:

$$\mathscr{Y} = \coprod_{\lambda \in S_n^\wedge} \{v_T\}_\lambda$$

Recall that the YJM-elements consist a set of generators of the Gelfand-Tsetlin algebra, which is the algebra of all operators diagonal in the Gelfand-Tsetlin basis. Hence the Young basis is a common eigenbasis of the YJM-elements, which enables us to make the following definition:

Definition 1.4. For any element $v \in \mathscr{Y}$, the **weight** of v, denoted as $\alpha(v)$, is the vector

$$\alpha(v) = (a_1, \ldots, a_n) \in \mathbb{C}^n$$

where a_i is the eigenvalue of X_i on v.

Since each $v \in \Upsilon$ is determined uniquely by the eigenvalues of the elements of $GZ(n)$ on v, the map $v \mapsto \alpha(v)$ is a one-to-one correspondence, the inverse of which we will denote as $\alpha \mapsto v_\alpha$.

Definition 1.5. The spectrum of the YJM-elements, or ***Spec(n)*** is defined as

$$Spec(n) = \{\alpha(v) \mid v \in \Upsilon\}$$

Also, define a equivalence relation $\sim$ as

$$\alpha \sim \beta, \quad \alpha, \beta \in Spec(n)$$

if v_α and v_β belong to the same irreducible representation of S_n.

Clearly there is a natural bijection between $Spec(n)$ and the Young basis $\mathscr{Y}$, as well as $Spec(n)/\sim$ and the set of irreducible representations $S_n^\wedge$.

1.2 The Plan

Under this context, our plan to analyze the irreducible representations of S_n is as follows:

1. Describe the set $Spec(n)$
2. Describe the equivalence relation $\sim$
3. Calculate the matrix elements in the Young basis
4. Calculate the characters of irreducible representations

Recall the following result proved in Ryan's talk:

Proposition 1.6. Let

$$\alpha = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \in Spec(n)$$

Then $a_1 = 0, a_i \in \mathbb{Z}$ and

1. $a_i \neq a_{i+1}$ for all i
2. if $a_{i+1} = a_i \pm 1$, then $s_i \cdot v_\alpha = \pm v_\alpha$
3. if $a_{i+1} \neq a_i \pm 1$, then

   $$\alpha' = s_i \cdot \alpha = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \in Spec(n)$$

   and $\alpha' \sim \alpha$.

2 Content Vectors

We aim to nearly complete the first two steps of the plan outlined above.

2.1 Motivation and Definition

It can be expected that the transpositions s_i in the third case of Proposition 1.6 will play an important role in further investigating what values $Spec(n)$ can take. We call them (i.e. s_i's such that $a_{i+1} \neq a_i \pm 1$) *admissible transpositions.* Indeed, the only hindrance of transporting the value of a_i to other positions by these transpositions are the entries that takes the values $a_i \pm 1$ (if they exist), and one can be motivated to inspect them. Under this motivation we state the following definition:

Definition 2.1. A vector $\alpha = (a_1, \ldots, a_n) \in \mathbb{C}^n$ is a **content vector** if α satisfies the following conditions:

1. $a_1 = 0$
2. for all $q > 1$, if $a_q > 0$, then $a_i = a_q - 1$ for some $i < q$; and if $a_q < 0$, then $a_i = a_q + 1$ for some $i < q$
3. if $a_p = a_q = a$ for some $p < q$, then

   $$\{a - 1, a + 1\} \subset \{a_{p+1}, \ldots, a_{q-1}\}$$

Also, define $\boldsymbol{Cont(n)}$ as the set of content vectors of length n, with an equivalence relation $\approx$ defined as

$$\alpha \approx \beta, \quad \alpha, \beta \in Cont(n)$$

if α is a permutation of β.[1]

Remark. It follows from easy induction that $Cont(n) \subset \mathbb{Z}^n$.

2.2 The Theorem

As wanted, it turns out that $Spec(n)$ is a subset of $Cont(n)$. To prove this, we need the following lemma.

Lemma 2.2. *Let $\alpha = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $a_i = a_{i+2} = a_{i+1} \mp 1$ for some i — that is, α contains a fragment of the form $(a, a \pm 1, a)$. Then $\alpha \notin Spec(n)$.*

Proof. Assume $\alpha \in Spec(n)$. By Proposition 1.6, $s_i v_\alpha = \pm v_\alpha$ and $s_{i+1} v_\alpha = \mp v_\alpha$. This implies $s_i s_{i+1} s_i v_\alpha = \mp v_\alpha$ and $s_{i+1} s_i s_{i+1} v_\alpha = \pm v_\alpha$, contradicting the identity $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.[2] $\square$

Now we proceed to prove the main theorem.

Theorem 2.3. *$Spec(n) \subset Cont(n)$.*

Proof. Let $\alpha = (a_1, \ldots, a_n) \in Spec(n)$. Denote the three conditions in Definition 2.1 as (1), (2), (3), and we show α satisfies each of these conditions. Since $X_1 = 0$, we have $a_1 = 0$, and (1) is true.

We prove that (2) and (3) hold for α by induction on n. The case $n = 1$ and $n = 2$ are straightforward, so let $n \geq 3$ and assume the induction hypothesis that the statement in true for $n - 1$. Since $(a_1, \ldots, a_{n-1}) \in Spec(n - 1)$, it suffices to prove when $q = n$ in both conditions.

Suppose $a_n > 0$. If $a_{n-1} = a_n - 1$ then (2) immediately follows. If $a_{n-1} = a_n + 1$, there exists $i < n - 1$ such that $a_i = a_{n-1} - 1 = a_n$. Using induction hypothesis again, there exists $j < i$ such that $a_j = a_i - 1 = a_n - 1$. If $a_{n-1} \neq a_n + 1$, then $(a_1, \ldots, a_{n-2}, a_n, a_{n-1}) \in Spec(n)$, and applying the induction hypothesis to $(a_1, \ldots, a_{n-2}, a_n) \in Spec(n - 1)$ shows that there exists some $i < n - 1$ that $a_i = a_n - 1$, so in all cases (2) is true. The case $a_n < 0$ can be proven exactly the same way, except that all the signs are reversed.

Now assume that $a_p = a_n = a$ for some $p < n$, and that $a-1 \notin \{a_{p+1}, \ldots, a_{n-1}\}$. We may assume that p is taken to be the largest possible index, so that $a \notin \{a_{p+1}, \ldots, a_{n-1}\}$. If $a+1$ occurs twice or more times in the set $\{a_{p+1}, \ldots, a_{n-1}\}$, a should also occur in it by induction hypothesis, so $a + 1$ occurs at most once in $\{a_{p+1}, \ldots, a_{n-1}\}$. Then $\{a_{p+1}, \ldots, a_{n-1}\}$ only contains the numbers different

[1] In the reference text, this equivalence relation $\approx$ is defined as $\alpha \approx \beta$ if β is an *admissible* permutation (i.e. the product of finite number of admissible transpositions) of α, and this is a mistake. We will soon see that these two definitions are compatible.

[2] Since $s_i s_{i+1}$ is a 3-cycle, $(s_i s_{i+1})^3 = \mathrm{id}$, so $s_i s_{i+1} s_i = {s_{i+1}}^{-1} {s_i}^{-1} {s_{i+1}}^{-1} = s_{i+1} s_i s_{i+1}$.

from $a-1, a, a+1$ with at most one possible exception that is equal to $a+1$. Hence, by applying a finite number of admissible transpositions, we obtian an element in $Spec(n)$ that contains a fragment (a, a) or $(a, a+1, a)$, which are both impossible by Proposition 1.6 and Lemma 2.2. So $a-1 \in \{a_{p+1}, \ldots, a_{n-1}\}$, and $a+1 \in \{a_{p+1}, \ldots, a_{n-1}\}$ can be proven exactly the same way except that all the signs are reversed. $\square$

3 The Young Graph

We now introduce a combinatorial object that visualizes the information of $Cont(n)$ (and thus helps us to work on it more easily).

3.1 Definitions

Definition 3.1. A **Young diagram** is a finite stack of boxes arranged in rows and columns in the way so that the first cell of each row lies in the first column and the lengths of rows are in non-increasing order.

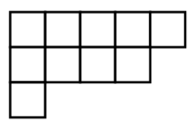

Figure 1: An example of a Young diagram.

We often denote the Young diagram as a list of the number of boxes in each row. For example, the above Young diagram is denoted $(5, 4, 1)$. In this sense, the set of Young diagrams with n boxes is in a natural bijection with the set of partitions of n.

Definition 3.2. A **Young graph** $\mathbb{Y}$ is a simple directed graph such that its vertices are the Young diagrams and two vertices ν, η are joined by a directed edge if $\nu \subset \eta$ and η/ν is a single box, in which case we write $\nu \nearrow \eta$.

The first five layers (excluding the empty set) of the Young graph are shown in the next page.

Definition 3.3. Given a box $\square \in \nu$ in a Young diagram, the number

$$c(\square) = (x\text{-coordinate of } \square) - (y\text{-coordinate of } \square)$$

is called the **content** of $\square$. Here, the coordinates are defined so that the boxes in the ith column have x-coordinates equal to $i-1$ and the boxes in the jth row have y-coordinates equal to $j-1$.

Definition 3.4. A **Young tableau** or **standard tableau** is a path in $\mathbb{Y}$ from $\emptyset$ to a certain vertice (i.e. a Young diagram). We denote $\boldsymbol{Tab(\nu)}$ the set of

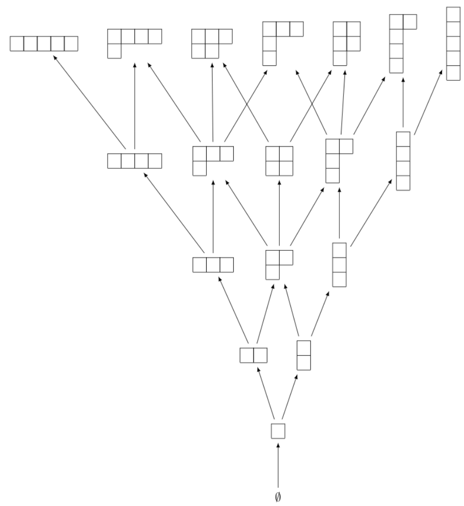

Figure 2: The Young graph.

Young tableaux from $\emptyset$ to ν. Also, let $Tab(n)$ the set of Young tableaux from $\emptyset$ to a Young diagram with n boxes, that is,

$$Tab(n) = \bigcup_{|\nu|=n} Tab(\nu)$$

.

A convenient way to represent a path $T \in Tab(\nu)$

$$\emptyset = \nu_0 \nearrow \cdots \nearrow \nu_n = \nu$$

is to label the boxes $\nu_1/\nu_0, \ldots, \nu_n/\nu_{n-1}$ of ν_n by numbers $1, \ldots, n$, respectively.[3]

3.2 Application to Analysis of $Spec(n)$

The following proposition writes the information of content vectors in terms of the notions related to the Young graph.

Proposition 3.5. Let

$$T = \nu_0 \nearrow \cdots \nearrow \nu_n \in Tab(n).$$

The mapping

$$T \mapsto (c(\nu_1/\nu_0), \ldots, c(\nu_n/\nu_{n-1}))$$

[3]Those who are familiar with the notion of Young tableau as an arbitrary labeling of n boxes in the Young diagram by $1, \ldots, n$ might be feeling a bit awkward, since our definition only induces the labelings such that the entries in each row and each column are increasing. Such tableau that satisfies this condition is called *standard* in the general context (as the alternative terminology of Definition 3.4 shows). In this particular talk, every Young tableau we'll discuss will be considered standard.

is a bijection between $Tab(n)$ and $Cont(n)$. Also, for any $\alpha, \beta \in Cont(n)$, we have $\alpha \approx \beta$ if and only if the corresponding paths in this bijection have the same end (i.e. they are the Young tableaux of the same Young diagram).

Proof. Take arbitrary $T = \nu_0 \nearrow \cdots \nearrow \nu_n \in Tab(n)$ and for $i \in \{1, \cdots, n\}$, denote the box in ν_i/ν_{i-1} as (x_i, y_i), where x_i and y_i are x-coordinate and y-coordinate of the box, respectively. Note that for every $i \in \{1, \cdots, n\}$ and a nonnegative pair of integers (p, q) such that $(p, q) < (x_i, y_i)$ (i.e. $p \leq x_i$, $q \leq y_i$, and $(p, q) \neq (x_i, y_i)$), there exists a positive integer $j < i$ such that $(x_j, y_j) = (p, q)$. In fact, the sequence of sets of boxes $\emptyset = \eta_0 \subset \eta_1 \subset \cdots \subset \eta_n$ such that $\eta_i/\eta_{i-1} = \{(u_i, v_i)\}$ gives a Young tableau if and only if for every $i \in \{1, \cdots, n\}$ and a nonnegative pair of integers (p, q) such that $(p, q) < (u_i, v_i)$, there exists a positive integer $j < i$ such that $(u_j, v_j) = (p, q)$. $\cdots$ $(*)$

First we prove that T is indeed mapped to a content vector. Denote the three conditions in Definition 2.1 as (1), (2), (3). Since $c(x_1, y_1) = c(0, 0) = 0 - 0 = 0$, the vector satisfies (1). For (2), if $c(x_i, y_i) = x_i - y_i > 0$, then $x_i \geq 1$, so for some $j < i$, $(x_j, y_j) = (x_i - 1, y_i)$ and $c(x_j, y_j) = x_j - y_j = x_i - y_i - 1 = c(x_i, y_i) - 1$; the case $c(x_i, y_i) < 0$ can be proved similarly. For (3), if $c(x_p, y_p) = c(x_q, y_q) = a$ for some $p < q$, it is straightforward that $x_p \leq x_q - 1$ and $y_p \leq y_q - 1$, so $(x_p, y_p) < (x_q - 1, y_q) < (x_q, y_q)$ and $(x_p, y_p) < (x_q, y_q - 1) < (x_q, y_q)$. Hence there exists $k, l \in \{p+1, \cdots, q-1\}$ such that $(x_k, y_k) = (x_q - 1, y_q)$, $(x_l, y_l) = (x_q, y_q - 1)$, and so $c(x_k, y_k) = a - 1$, $c(x_l, y_l) = a + 1$.

Now we prove that for any content vector $\alpha = (a_1, \cdots, a_n) \in Cont(n)$, there exists a unique Young tableau that maps to α by induction on n. The case $n = 1$ is trivial, so assume $n \geq 2$, and that the statement is true for $n - 1$. By the induction hypothesis, $\nu_0, \nu_1, \cdots, \nu_{n-1}$ are uniquely determined (under the assumption that such Young tableau exists), so it suffices to show that we can choose a unique (x_n, y_n) which induces a Young tableau and satisfies $c(x_n, y_n) = a_n$. If such (x_n, y_n) exists, obviously it should be the smallest pair of nonnegative integers (z, w) such that $z - w = a_n$ and $(z, w) \notin \nu_{n-1}$. I now claim this pair induces a Young tableau. By $(*)$, it suffices to prove that for any nonnegative pair of integers (p, q) such that $(p, q) < (z, w)$, $(x_j, y_j) = (p, q)$ for some $j < n$. If $z \geq 1$ and $w \geq 1$, by our construction of (z, w), there exists some $i < n$ such that $(x_i, y_i) = (z - 1, w - 1)$. Using condition (3), there exists $k, l \in \{i+1, \cdots, n-1\}$ such that $c(x_k, y_k) = a_n - 1 = z - w - 1$ and $c(x_l, y_l) = a_n + 1 = z - w + 1$. Since $(x_i, y_i) < (x_k, y_k) < (x_n, y_n)$ and $(x_i, y_i) < (x_l, y_l) < (x_n, y_n)$, it must follow that $(x_k, y_k) = (z - 1, w)$ and $(x_l, y_l) = (z, w - 1)$. Hence we are done by induction hypothesis, since either $(p, q) \leq (z - 1, w)$ or $(p, q) \leq (z, w - 1)$. If $z = 0$, then $p = 0$, $q \leq w - 1$. By condition (2), since $a_n = -w < 0$, there exists some $i < n$ such that $c(x_i, y_i) = a_n + 1 = -w + 1$. It is easy to deduce that $(0, q) \leq (x_i, y_i)$, so we are done by induction hypothesis. The case $w = 0$ can be settled in the same way.

Finally, we prove the second part of the proposition. If $\alpha, \beta \in Cont(n)$ satisfy $\alpha \approx \beta$ (i.e. α is a permutation of β), let S the multiset of entries of α. Obviously S is also the multiset of entries of β. Suppose t is any integer, and appears n times in S. By the proof of first part of the proposition above, the Young diagrams of the corresponding Young tableaux T_α, T_β contain the smallest n pairs (z_i, w_i) of nonnegative integers such that $c(z_i, w_i) = t$, and doesn't contain any other pair (z', w') such that $c(z', w') = t$. Since this is true for every integer t, the diagrams should be equal. Conversely, if the corresponding final diagrams are equal, $\alpha \approx \beta$ since the multiset of entries of a content vector is equal to the multiset of values of the content of the boxes in the corresponding Young diagram. $\square$

Now consider the transpositions on the Young tableau that switches the labels of two boxes whose labels are consecutive integers. This operation produces a Young tableau if and only if the two boxes are located at different rows and different columns, which is equivalent to the fact that the content values of the two boxes doesn't have a difference ± 1 (due to the fact that the two boxes have consecutive labels). In the bijection above, this corresponds exactly to the admissible transpositions in the content vector. Hence admissible transpositions are transpositions in the Young tableau that switches the boxes with consecutive labels and preserves the Young tableau structure.

Proposition 3.6. Any two Young tableaux $T_1, T_2 \in Tab(\nu)$ with diagram ν can be obtained from each other by a sequence of admissible transpositions. In other words, if $\alpha, \beta \in Cont(n)$ and $\alpha \approx \beta$, then β can be obtained from α by a sequence of admissible transpositions.

Proof. Induct on n; base case $n = 1$ is easy, so assume $n \geq 2$ and the statement is true for $n - 1$. It suffices to show that we can transform any Young tableau $T \in Tab(\nu)$, $\nu = (\nu_1, \cdots, \nu_k)$ to the tableau T_0 with the same diagram and labeled with horizontal monotone numeration (i.e. the box $(0, 0)$ is labeled 1, $(1, 0)$ is labeled 2, $\cdots$, $(\nu_1 - 1, 0)$ is labeled ν_1, $(0, 1)$ is labeled $\nu_1 + 1$, $\cdots$, $(\nu_2 - 1, 1)$ is labeled $\nu_1 + \nu_2$, $(0, 2)$ is labeled $\nu_1 + \nu_2 + 1$, $\cdots$). Consider the box $B = (\nu_k - 1, k - 1)$ — the last box of the last row or ν — and let i the label of this box in T. Since all other boxes that share the same row or column with B have both coordinates smaller than or equal to those of B, they are labeled with an integer less than i. Hence $(i, i + 1)$ is an admissible transposition, and we obtain another tableau after swtiching the labels i and $i+1$ in the corresponding boxes. After this operation, B is labeled $i+1$, and $(i+1, i+2)$ is an admissible transposition in this new tableau. Repeating this $n - i$ times, we obtain a tableau with B labeled as n. Since the label of B in T_0 is also n, by inductive hypothesis T_0 can be obtained from this tableau by a sequence of admissible transpositions. $\square$

Corollary 3.7. If $\alpha \in Spec(n)$, $\beta \in Cont(n)$, and $\alpha \approx \beta$, then $\beta \in Spec(n)$ and $\alpha \sim \beta$.

Proof. By Proposition 3.6, β can be obtained from α by a sequence of admissible transpositions. Hence by (3) of Proposition 1.6 β is in the same equivalence class of α in $Spec(n)$. $\square$

4 What's Next?

It turns out that $Spec(n) = Cont(n)$, the corresponding equivalence relations $\sim$ and $\approx$ coincide, and the Young graph $\mathbb{Y}$ is the branching graph of the symmetric groups when each Young diagram is looked upon as its corresponding irreducible representation (We are almost done with proving this!). This completes the task of describing the set $Spec(n)$ and the equivalence relation $\sim$ in our plan. We will go on to obtain an explicit model of representations of S_n and sketch the derivation of the formula for their characters.